PHYSICS 715

Problem Set 11 **Due Friday, April 28, 2006**

Reading: Landau and Lifshitz, Secs. 62, 64-68, 71-72,

LD 36: The chemical potential and vapor pressure of a Debye solid.

(a) Determine the chemical potential μ_s of a three-dimensional Debye solid with a common speed of propagation v for transverse and longitudinal waves. The solid contains N atoms and has an intrinsic binding energy ϵ_0 per atom (*not* per mode). [Hint: use the canonical approach, and calculate μ_s using the thermodynamic relation $\mu_s = (\partial F/\partial N)_{T,V}$. Be sure to include the zero point energies of oscillation. It is simplest not to introduce the Debye function or the Debye temperature.]

(b) Obtain an expression for the vapor pressure P of a gas of the same atoms in equilibrium with the solid. Plot P (in atmospheres, on a scale of powers of 10) as a function of T for $\epsilon_0 = 1.4$ eV, $v = 3000$ m/s, $n = 6 \times 10^{28}$ atoms/m^3, and atomic weight 60. [Note: the Debye model only makes sense at temperatures below the melting point of the solid. This can be estimated roughly as $kT_{\rm mp} \approx mv_\ell^2/100$, corresponding to thermal displacements of atoms from equilibrium of $\sim 1/10$ of the interatomic spacing. Cover the region from ≈ 200 K below $T_{\rm mp}$ to $T_{\rm mp}$ in your plot.]

LD 37: Adiabatic relations for Fermi and Bose gases and the properties of the early Universe.

The total mass density of the Universe is observed to equal the critical density $\rho_{\rm c} \approx 0.9 \times 10^{-26}$ kg/m^3. Of this, $\approx 5\%$ is now associated with baryons and electrons and 25% with dark (nonluminous) matter. The latter is thought to consist mainly of massive nonrelativistic exotic particles. The total matter density is $\rho_{\rm m} = 2.7 \times 10^{-27}$ kg/m^3. The remaining 70% of $\rho_{\rm c}$ arises almost entirely from dark or vacuum energy $\rho_{\rm vac}$, with a little "radiative" energy density $\rho_{\rm rad}$ from photons and neutrinos. The temperature of the cosmic blackbody radiation (a relic of the time at which the radiation and matter were in equilibrium) is 2.75 K. $\rho_\gamma \equiv E_\gamma/c^2$ where E_γ is the photon energy density.

(a) Because of the Hubble expansion, the temperature and energy density in the Universe are constantly decreasing. Estimate the temperature T at the time at which the energy densities of matter and radiation were equal, and determine the ratio $R(T)/R_0$ of the radius of the Universe at temperature T to its present radius R_0. The expansion of the universe may be treated as adiabatic, with the matter, radiation, and dark energy decoupled or noninteracting.

(b) The Universe, determined to be flat, expands according to the Friedman equation

$$\left(\frac{\dot{R}}{R}(t)\right)^2 = \frac{8\pi G}{3}\left[\rho_{\rm m} + \rho_{\rm rad} + \rho_{\rm vac}\right](t) \equiv H^2(t)$$

where $H(t) = (8\pi G\rho/3)^{1/2}$ is the Hubble parameter. Determine the relation between time and temperature in the present era where most of the mass density comes from nonrelativistic particles and the constant vacuum energy, and in the "radiation-dominated" era when most of the energy density is associated with photons and ultrarelativistic particles $\rho_{\rm rad}$. Use the results to estimate

(i) the time in years since the energy densities of matter and radiation were equal (include the effects of the matter and vacuum energy only); and

(ii) the age of the universe at the time of primordial nucleosynthesis when T was $\approx 10^9$ K.

Assume that the only relativistic particles are photons and the three known neutrinos and antineutrinos, each with $g_\nu = 1$.

Determine the mass density E_γ/c^2 (in gm/cm^3) in the blackbody radiation, and the number density of the baryons at the time of nucleosynthesis. Should we understand the physics for these conditions? Explain.

LD 38: The Thomas-Fermi model for atoms: a self-consistent mean field model.

The Thomas-Fermi statistical model of the atom describes the electrons in the atom as a degenerate Fermi gas confined in a (self-consistent) potential $\phi(r)$. Suppose ϕ is of the form

$$\phi(r) = -\frac{Ze^2}{r}\chi\left(\frac{r}{R}\right), \quad R = \text{constant},$$

and that the binding energy of the last electron is approximately zero (this is a reasonable assumption for heavy atoms, for which the binding energies of the core electrons are very large compared to the binding energy of the last electron).

(a) Determine the electron density distribution $n(r)$ and its normalization assuming $\phi(r)$ to be known, and show that $R \propto Z^{-1/3}$ whatever the form of χ.

(b) Determine R/a_0 for the (excellent) approximation $\chi_0 = \left(1+\frac{r}{R}\right)^{-2}$, $a_0 = \hbar^2/m_e e^2$ = Bohr radius, and compare χ_0 numerically with the exact Thomas-Fermi function $\chi(x)$ [see, *e.g.*, Condon and Shortley, *Theory of Atomic Spectra*, p. 337. Express r/R in terms of the Thomas-Fermi variable $x = 2\left(\frac{4}{3\pi}\right)^{2/3} Z^{1/3}\frac{r}{a_0}$.

(c) Find the nonlinear differential equation satisfied by $\chi(x)$ (the Thomas-Fermi equation) by requiring that the electrostatic potential $e^{-1}\phi(r)$ be related to the charge density in the atom through Poisson's equation. What are the boundary conditions on χ? Explain. [Hint: Don't forget the nuclear charge density. Use $\nabla^2 = r^{-2}\frac{d}{dr}r^2\frac{d}{dr}$ for a spherically symmetric system with $r \neq 0$, and recall that $\nabla^2\frac{1}{r} = -4\pi\delta^3(\vec{r})$.]

For a thorough discussion of the Thomas-Fermi model, its extensions, and applications to a number of interesting problems (proof of the stability of matter, determination of the size and binding energies of large atoms, the application of similar ideas to white dwarfs, neutron stars, etc.), see L. Spruch, Rev. Mod. Phys. **63**, 151 (1991).