PHYSICS 715

Problem Set 12 **Due Friday, May 5, 2006**

Reading: Landau and Lifshitz, Secs. 148-150, 152-153
Huang, Secs. 14.1-14.4, 16.1-16.4 (suggested)

FINAL EXAM MONDAY, MAY 8, 12:25 pm

LD 39: Bose-Einstein condensation in an atomic trap.

Bose-Einstein condensation was observed directly in 1995 in dilute atomic gases confined in magnetic traps (see M. H. Anderson et al., Science **269**, 198 (1995); C. C. Bradley et al., Phys. Rev. Lett. **75**, 1687 (1995); K. B. Davis et al., Phys. Rev. Lett. **75**, 3969 (1995)). The traps are approximately harmonic with characteristic oscillation frequencies ν of about 150 Hz. The critical temperature T_c at which the condensation starts varies with the experimental conditions, but is typically $\approx$ 150 nK.

(a) A mean field model for an ideal Bose gas in an external potential $V(\mathbf{r})$ describes the gas in a volume d^3r at a point $\mathbf{r}$ as being in local equilibrium and having a local chemical potential $\mu(\mathbf{r}) = \mu - V(\mathbf{r})$, where μ is the constant chemical potential for the entire system. When is this a reasonable approximation? The number distribution of the particles in the volume d^3r is given in this picture by a standard Bose number distribution with μ replaced by $\mu(\mathbf{r})$, and the usual volume factor replaced by d^3r:

$$dN = \frac{1}{e^{\beta(E(\mathbf{p})-\mu(\mathbf{r})} - 1}\frac{d^3p\, d^3r}{h^3} = \frac{1}{e^{\beta(H(\mathbf{p},\mathbf{r}))-\mu} - 1}\frac{d^3p\, d^3r}{h^3}.$$

Integrate over all positions and momenta to obtain an expression for the maximum number of particles N that can be accomodated in a harmonic trap with $V = \frac{1}{2}m\omega^2\mathbf{r}^2$, and use the result to determine the critical temperature for Bose-Einstein condensation in terms of N and the parameters in H. Evaluate T_c for $N = 4\times10^4$ ^{87}Rb atoms in a 150 Hz trap ($5S_{1/2}$ electron and a nucleus with spin $s = 3/2$ in a total angular momentum state with $F = 2,\ m_F = 2$). Compare the result to that in the first reference above. [Hint: the integral can be reduced to a standard form by introducing scaled variables $r' = r/\lambda,\ p' = \lambda p,\ \lambda = 1/\sqrt{m\omega}$, introducing the the 6-dimensional vector $\mathbf{x} = (\mathbf{r}', \mathbf{p}')$, writing the volume element as d^6x, and changing to x^2 as the integration variable after performing the angular integration.]

Determine the number N_0 of the atoms in the ground state of the system as a function of T/T_c, and plot of the ratio N_0/N versus T/T_c.

(b) The ground state wave function for an isotropic oscillator in three dimensions is

$$\psi_0(r) = \frac{1}{\pi^{3/4} r_0^{3/2}} e^{-r^2/2r_0^2}, \qquad r_0 = \sqrt{\frac{\hbar}{m\omega}}.$$

Determine the scaled number distribution $\frac{1}{N}\frac{dN}{d^3(r/r_0)}$ of atoms in the trap and plot it as a function of r/r_0 for $N_0/N = 0, 0.1, 0.3, 1.0$ and $0 \leq r/r_0 \leq 10$. Evaluate d^3N/dr^3 at $r = 0$. Is this a high density? Explain. Compare your results qualitatively with those in the references above. [Hint: the momentum integral which appears must be evaluated numerically. Express the integrand in terms of $\hbar\omega/kT_c$, T/T_c, and r/r_0 to see its structure before doing the integration. Note that for a given T_c, $N = N_c$.]

LD 40: The Bragg-Williams approximation for the 3-dimensional Ising model as a Landau-Ginzburg mean field theory.

Mean field theory (or the Bragg-Williams approximation) gives the expression

$$m = \mu_0 \tanh\left[\frac{\mu_0 \mathcal{H}}{kT} + \frac{T_c}{T}\frac{m}{\mu_0}\right]$$

for the magnetic moment per spin in an Ising model with particles with magnetic moment μ_0. $\mathcal{H}$ is the applied magnetic field. The total magnetization per unit volume is $M = nm$, where n is the density of spins.

(a) Show that the magnetic Gibbs function has the form assumed in the Landau-Ginzburg mean field approach when calculated to order m^4 with $\mathcal{H}$ retained as a free variable. [Hint: rewrite the relation above as an equation for $\mathcal{H}$, expand the inverse hyperbolic tangent which appears to order m^3, and determine F_M by integrating the relation $\mathcal{H} = \frac{\partial F}{\partial M}$. $G_M = F - M\mathcal{H}$. *Do not* eliminate $\mathcal{H}$ in G_M.]

(b) Show that T_c is in fact the critical temperature at which spontaneous magnetization appears at $\mathcal{H} = 0$ in the mean field theory. Determine the temperature dependence of the magnetization and the magnetic susceptibility $\chi = \partial M/\partial\mathcal{H}$ for $\mathcal{H} \to 0$ to lowest order in $|T_c - T|$ for $T > T_c$ and for $T < T_c$, and find the critical exponents β and γ. [Hint: minimize G_M at fixed $\mathcal{H}$ to determine M. Recall that, in the Landau approach, one is dealing with a Taylor series expansion in T as well as M, and replace T by T_c wherever it does not appear in the difference $|T - T_c|$. Show that your solution for M actually minimizes G_M in the two temperature regions.]

(c) Plot separate figures giving the approximate expressions for $M/n\mu_0$ and $\chi kT_c/n\mu_0^2$ for $0 \leq T/T_c \leq 2$ and values in the the ranges $0 \leq M/n\mu_0 \leq 2$ and $0 \leq \chi kT_c/n\mu_0^2 \leq 4$. Include on the same figures the exact results for these quantities obtained by solving the equation above and the corresponding equation for $dM/d\mathcal{H}$ (evaluated for $\mathcal{H} = 0$). [Hint: To find the nontrivial solution for M numerically for $T < T_c$, find the zero of $1 - \frac{1}{x}\tanh\frac{T_c}{T}x$, $x = m/\mu_0$.]

LD 41: First order phase transitions in the Landau theory.

Consider a mean field theory with order parameter m and Landau free energy

$$G = \frac{1}{2}am^2 - \frac{1}{3}bm^3 + \frac{1}{4}cm^4, \qquad a, b, c > 0.$$

Sketch the possible shapes for $G(m)$ as b is increased from zero with a and c fixed, and show that a first order phase transition occurs for b equal to a critical value b_c. Determine b_c and the change in the equilibrium value of m at the transition. [Hint: The scaling $m = (2a/c)^{1/2}m'$, $G = (a^2/c)G'$ is useful for seeing the structure of G. Make your sketches for m' small. To find b_c, determine when G has two quadratic minima with $G = 0$, but at different values of m.]

LD 42: Critical exponents from a Landau free energy.

The Landau free energy for a hypothetical system is of the form

$$F(t, m, h) = -hm + \frac{1}{4}atm^4 + \frac{1}{6}bm^6, \qquad a, b > 0,$$

where m is the order parameter, $t = (T - T_c)/T_c$, and the external field h is the variable conjugate to m. Show that there is a "second order" phase transition at $T = T_c$ for $h = 0$. Determine the critical exponents $\alpha, \beta, \gamma, \delta$, and show directly that

$$\alpha + 2\beta + \gamma = 2, \qquad \delta = 1 + \frac{\gamma}{\beta}.$$

Show explicitly that your solution for m actually minimizes F for $t > 0$ and $t < 0$. Keep only the leading behavior in t for $t \to 0$. [Hint: Calculate the thermodynamic derivatives which determine S and $\chi = \left(\frac{\partial h}{\partial m}\right)_t^{-1}$, and evaluate them for the value of m that minimizes F. $C = T\frac{\partial S}{\partial T}$. To determine δ, minimize F on the critical isotherm $t = 0$ for $h \neq 0$.]