PHYSICS 715

Problem Set 2 **Due Friday, February 3, 2006**

Reading: Landau and Lifshitz, Chap. 2; Chap. 3, Secs. 28–31
Huang, Secs. 7.1–7.6 (suggested)

LD 4: Hawking radiation and the lifetime of black holes

(a) It was observed by J. Bekenstein, Phys. Rev. D **7**, 2333 (1973), that the entropy of matter falling into a black hole should increase the entropy of the black hole, and that the entropy of the hole should be proportional to its area. The resulting temperature of a black hole of mass M or energy Mc^2 is $T = \hbar c^3/8\pi kGM$, where G is Newton's constant.

Determine the entropy S assuming that $S = 0$ for a zero-mass black hole, and find the dependence of the area $A = 4SL_{Pl}^2/k$ on M. Here L_{Pl} is the Planck length, $L_{pl} = (\hbar G/c^3)^{1/2} \approx 1.6 \times 10^{-33}$ cm. Estimate the size of a black hole of solar mass, $M = 2 \times 10^{33}$ gm.

(b) A black hole with $T > 0$ will radiate photons (and neutrinos) with a thermal spectrum [S.W. Hawking, Nature **248**, 30 (1974); Comm. Math. Phys. **43**, 199 (1975); Phys. Rev. D **13**, 191 (1975)]. The power radiated per unit area in this "Hawking radiation" is given by the expression for blackbody radiation, $P = \sigma T^4$, up to a factor of order unity. Here σ is the Stefan-Boltzmann constant $\sigma = \pi^2 k^4/60\hbar^3 c^2$. Use this result to estimate how massive a black hole formed in the big bang must be if it is to have survived $\approx 13.7 \times 10^9$ yr to the present.

LD 5: Increase of entropy and heat flow.

Two large many-particle systems A and B with initial energies E_A^0 and E_B^0 are brought into contact. Assume that there is no mixing of the constituents (for example, A and B could be solids), but that energy can be transferred between A and B. State your physical assumptions, and use the Boltzmann definition of entropy in terms of Γ_{AB} to show

(i) that entropy increases when the two systems are combined, i.e., that S_{AB} is greater than or equal to the total entropy of the separate systems once AB reaches equilibrium,

$$S_{AB}(E) \geq S_A(E_A^0) + S_B(E_B^0), \quad E = E_A + E_B;$$

(ii) that energy flows from the hotter to the colder system. [Hint: use $\Delta S \geq 0$, the extensive property of the entropy, and appropriate Taylor series expansions to show that

$$\left(\frac{1}{T_A^0} - \frac{1}{T_B^0}\right)(\bar{E}_A - E_A^0) \geq 0,$$

for $\bar{E}_A - E_A^0$ small and discuss the consequences.]

LD 6: Excluded volumes and the equation of state for a hard-sphere gas.

For "hard" molecules of finite size, the factor V^N in the statistical weight $\Gamma(E, N, V)$ is changed. The first molecule can be anyplace in the volume V. Once its location is fixed, the center of the second molecule must be at least a distance $2R$ away, where R is the molecular radius. That is, the second molecule can be anyplace in a reduced volume $V - \beta$ where $\beta = 8v$ with v the volume of a single molecule, the third can be anyplace in a volume $V - 2\beta$, ... (This neglects possible simultaneous overlaps of $3, 4, \ldots, N$ molecules.) Evaluate $\int d^3x_1 \cdots d^3x_N$ in this approximation and determine the equation of state of the hard sphere gas from the relation $P = T\,(\partial S/\partial V)_E$ from the second law.

[Hint: it will be useful to factor $(8v)^N$ out of your expression, write the remainder as a ratio of factorials, and use Stirling's approximation.]

Expand your expression for P for a dilute gas, $8Nv \ll V$, and relate v to the constant b in the phenomenological van der Waals equation of state (Landau and Lifshitz, §76).

LD 7: Energy distribution function and fluctuations for the ideal gas.

The energy distribution function $W(E)$ for the canonical distribution is defined as

$$W(E) = \int' \delta(E - H) e^{-H/kT} \frac{d^3q_1 \ldots d^3q_N d^3p_1 \ldots d^3p_N}{h^{3N} Z_N},$$

where E is the total energy of the system, and $W(E)dE$ is the probability of finding the system in the energy interval dE at E. Z_N is the canonical partition function.

(a) Determine $W(E)$ for a system of N *indistinguishable* noninteracting particles in a box of volume V. The particles have the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} \qquad .$$

[Hint: use spherical coordinates in the 3N-dimensional momentum space. Recall that $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ defines the gamma function.]

(b) Calculate the most probable total energy E_m for the system (the energy for which $W(E)$ has its maximum value), the average energy $\bar{E}$, and their fractional deviation $|E_m - \bar{E}|/\bar{E}$. Comment on the N dependence.

(c) Show by direct calculation using $W(E)$ that the mean square fluctuation $\langle (E - \bar{E})^2 \rangle$ of E about $\bar{E}$ is equal to $kT^2 C_V$, where C_V is the specific heat at constant volume given by elementary kinetic theory, $C_V = \frac{3}{2}Nk$. [This is an example of the general *fluctuation-dissipation* theorem, which relates *fluctuations* in a variable to the *linear response* of the system to changes in an associated quantity.]