

energy by a suitable constant,<sup>7,8</sup> or by allowing the particles to interact at a greater distance than they really do. With (2b) we found that multiplying the potentials by 1.36 yields a function for the sum of all energies which has a minimum value of  $-246 \text{ mc}^2$  at  $\sigma=0.80$  (taking  $\tau=\sigma$  since very little improvement is obtained from varying both  $\tau$  and  $\sigma$  independently). The second method (method (i)) is believed to be somewhat more reliable. With (2b) the radius of action must be increased by the factor 1.3 to make the minimum value of  $E(\text{O}^{16})$  agree with experiment. The minimum is assumed at  $\sigma=1.01$  (again taking  $\sigma=\tau$  for the same reason as before). To obtain the numerical results discussed in the text we use the values

$$\begin{aligned} L_{\nu\pi} &= 15.74, & K_{\nu\pi} &= 1.41, \\ L_{\nu\nu} &= 10.05, & K_{\nu\nu} &= 0.90, \\ L_e &= 0.892, & K_e &= 0.055, & \text{Method (i).} & (24) \\ CE(s) &= 1.09, & CE(ps) &= 1.63, \\ & & 3(1-g)D(\sigma, \sigma)A_{\nu\pi} &= 14.95, \\ & & 6(1-g)B(\sigma)A_{\nu\pi} &= 120.00. \end{aligned}$$

In computing the kinetic and Coulomb energies  $\alpha$  is replaced by  $\alpha/(1.3)^2$ .

The straight-forward application of this procedure to the models (2a) and (2c) is not very satisfactory, because an increase in the strength of the potentials or in the radius of action which serves to fit the experimental  $\text{O}^{16}$  energy will give far too much energy to the lighter elements.

Finally it was found most satisfactory to make the calculations without modifying either the range or depth of the potentials. With  $\sigma=1.01/(1.3)^2$  the kinetic and Coulomb energies are exactly as in method (i). The other matrix elements have the values

$$\begin{aligned} L_{\nu\pi} &= 9.83, & K_{\nu\pi} &= 1.44, \\ L_{\nu\nu} &= 6.19, & K_{\nu\nu} &= 0.90, & \text{Method (ii).} & (25) \\ & & 3(1-g)D(\sigma, \sigma)A_{\nu\pi} &= 12.13, \\ & & 6(1-g)B(\sigma)A_{\nu\pi} &= 77.37. \end{aligned}$$

The correlation energy is introduced by adding to the total computed energy a linear function of the number of particles. Since the general linear function contains two parameters it is possible in this way to fit exactly the measured binding energies of two different nuclei, i.e.,  $\text{He}^4$  and  $\text{O}^{16}$ . Then the energies of all the others are uniquely determined.

## On the Consequences of the Symmetry of the Nuclear Hamiltonian on the Spectroscopy of Nuclei

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The structure of the multiplets of nuclear terms is investigated, using as first approximation a Hamiltonian which does not involve the ordinary spin and corresponds to equal forces between all nuclear constituents, protons and neutrons. The multiplets turn out to have a rather complicated structure, instead of the  $S$  of atomic spectroscopy, one has three quantum numbers  $S$ ,  $T$ ,  $Y$ . The second approximation can either introduce spin forces (method 2), or else can discriminate between protons and neutrons (method 3). The last approximation discriminates between protons and neutrons in method 2 and takes the spin forces into account in method 3. The method 2 is worked out schematically and is shown to explain qualitatively the table of stable nuclei to about  $\text{Mo}$ .

### 1

RECENT investigations<sup>1</sup> appear to show that the forces between all pairs of constituents of the nucleus are approximately equal. This makes it desirable to treat the protons and neutrons on an equal footing. A scheme for this was devised in his original paper by W. Heisen-

berg<sup>2</sup> who considered protons and neutrons as different states of the same particle. Heisenberg introduced a variable  $\tau$  which we shall call the isotopic spin, the value  $-1$  of this variable can be assigned to the proton state of the particle, the value  $+1$  to the neutron state. The assumption that the forces between all pairs of particles are equal is equivalent, then, to the assumption that they do not depend on  $\tau$  or that the Hamiltonian does not involve the isotopic spin.

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<sup>1</sup> M. A. Tuve, N. P. Heydenburg and L. R. Hafstad, *Phys. Rev.* **50**, 806 (1936); G. Breit, E. U. Condon and R. D. Present, *Phys. Rev.* **50**, 825 (1936).

<sup>2</sup> W. Heisenberg, *Zeits. f. Physik* **77**, 1 (1932).

In addition to this isotopic spin  $\tau$ , we must keep, of course, the ordinary spin variable  $s$  also;  $s$  also can assume the two values  $+1$  and  $-1$ . It has been pointed out lately<sup>3</sup> that the Pauli principle requires that the wave function

$$\Psi(r_1 s_1 \tau_1, r_2 s_2 \tau_2, \dots, r_n s_n \tau_n) \quad (1)$$

be antisymmetric with respect to the simultaneous interchange of Cartesian, spin and isotopic spin variables of any pair of heavy particles. This fact is quite analogous to the similar statement for ordinary spin.

Of course, if Eq. (1) is to represent the state of a given nucleus, say with  $n_P$  protons and  $n_N$  neutrons, it must vanish at every place where the sum of the  $\tau$ 's

$$\tau_1 + \tau_2 + \dots + \tau_n \neq n_N - n_P \quad (2)$$

is not equal to the "isotopic number" of this element. All wave functions which are finite for several sums of the  $\tau$ 's, refer to states which can be different elements with finite probabilities. No such states are known to be of any importance and the mathematical apparatus of the isotopic spin is, hence, somewhat redundant. It will turn out that it is very useful in spite of this.

In addition to the assumption of the approximate equality of forces between all pairs of particles, it appears to be a useful approximation to neglect the forces involving the ordinary spin. The Hamiltonian depends then on the space coordinates alone. By keeping both, one or none of these assumptions, one comes to four possible schemes;

(1) Take into account forces depending on space coordinates alone.

(2) Take into account forces depending on space and ordinary spin coordinates, assuming, however, interactions between all kinds of pairs to be equal.

(3) Neglect ordinary spin forces, take into account forces depending on space coordinates and isotopic spin, i.e., discriminate between proton-proton, proton-neutron and neutron-neutron interactions.

(4) Take all kinds of interaction into account.

The first is the roughest method, the last the most exact and it is probable that (2) is more accurate for light elements, (3) for heavy

elements. On the other hand, of course, one can obtain most results from symmetry considerations for 1, fewest for 4. Approximation (1) is identical with the "all orbital forces equal" model.<sup>4</sup>

The statement that an operator involves only one or another set of variables needs further amplification. As used in the ordinary theory of spectra, this expression means that the operator *can be written* in terms of these variables alone. It did not mean that it cannot be written in some other way as well. Thus, e.g., the interchange  $P$  of the space coordinates acts only on space coordinates, although it can be written by Dirac's identity,

$$P = -\frac{1}{2} - \frac{1}{2}(s_1 \cdot s_2)$$

entirely in terms of spin operators for antisymmetric functions. We shall keep this definition for the forces depending on Cartesian and ordinary spin coordinates for nuclei also.

The operators which involve  $\tau$  are, however, somewhat specialized to begin with. Using Heisenberg's notation for isotopic spin operators

$$\tau = \tau_z = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}, \quad \tau_x = \begin{vmatrix} 0 & i \\ -i & 0 \end{vmatrix}, \quad \tau_y = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad (4)$$

the conservation law for electric charge requires that all operators commute with

$$\tau_{z1} + \tau_{z2} + \dots + \tau_{zn} = n_N - n_P = 2T_z. \quad (3)$$

In addition to this, one hardly would say that

$$\tau_{z1}\tau_{z2} + \tau_{y1}\tau_{y2} + \tau_{x1}\tau_{x2} = -1 - 2PQ. \quad (5)$$

( $P$  interchange of space,  $Q$  interchange of spin coordinates) does not involve the Cartesian or spin coordinates, since Eq. (5) is a rather artificial expression,  $\tau_x$  and  $\tau_y$  having no immediate physical significance. We shall assume hence for approximation (3) only such operators which are equivalent to operators acting on the Cartesian coordinates alone, but in a different way for protons and neutrons. This is equivalent to using only operators involving the space coordinates and the  $\tau_z$ 's. If we do this, the results of method (3) must become equivalent to the usual theory (without  $\tau$ 's) which neglects

<sup>3</sup> J. H. Bartlett, Phys. Rev. **49**, 102 (1936); W. Elsasser, J. de phys. et rad. **7**, 312 (1936), and especially B. Cassen and E. U. Condon, Phys. Rev. **50**, 846 (1936).

<sup>4</sup> E. Feenberg and E. Wigner, Phys. Rev. This issue.

the spin. As a matter of fact, for approximation (3), the introduction of  $\tau$  is entirely useless and it is taken up here only in order to establish the transition from approximation (1) to (3).

2

The interaction in the electronic shells of atoms is a sum of terms containing two particles only and the momenta is no higher than the second power. The reason for the first is, that the interaction occurs through a field and this gives in first approximation only interaction between two particles. The reason that one can stop with the second power of the momenta is that these always enter in the combination  $p/mc$  which is a small quantity.

An advantage of introducing the variable  $\tau$  is<sup>2, 3</sup> that one can take over these assumptions to nuclei. If one does not use the variable  $\tau$  the interchange of two particles if expressed as a power series of the momenta is an infinite series<sup>5</sup>

$$\sum_{n_1 n_2 n_3} \frac{(x_2 - x_1)^{n_1} (y_2 - y_1)^{n_2} (z_2 - z_1)^{n_3}}{n_1! n_2! n_3!} \times \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right)^{n_1} \left( \frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2} \right)^{n_2} \left( \frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2} \right)^{n_3}.$$

However, it can be expressed by means of Dirac's identity also entirely without the momenta by means of Eq. (5). It must be admitted, however, that the spin cannot be considered to be small as in the atomic theory. We shall determine here all interaction forms between two particles which do not contain higher than first power terms of momenta<sup>6</sup> as far as the dependence on  $s$  and  $\tau$  goes. Nothing can be said, of course, on the dependence on the distance, and this factor will be omitted hence. It seems to be of lesser importance for the present.

The interaction must have spherical symmetry, depending on the differences of coordinates and momenta only, be invariant under inversion, substitution of  $-t$  for  $t$  and also be

<sup>5</sup> J. A. Wheeler, Phys. Rev. 50, 643 (1936).

<sup>6</sup> Some of these were given previously by Cassen and Condon, reference 3. The expressions given here are invariant only under Galilei transformations. G. Breit has shown that, in order to ensure relativistic invariance, correction terms must be added to the expressions derived here.

symmetric in the particles. The first requirements determine the dependence on  $s$ ,  $x$  and  $p$ . From the two triples of spin operators, one can form two invariants

$$(i) \quad 1; \text{ and } (i') \quad \frac{1}{2} + \frac{1}{2}(s_{x1}s_{x2} + s_{y1}s_{y2} + s_{z1}s_{z2}) = Q_{12}$$

three axial vectors with  $Z$  components

$$(v) \quad s_{z1} + s_{z2}; \quad s_{z1} - s_{z2}; \quad s_{x1}s_{y2} - s_{y1}s_{x2},$$

respectively, and one axial tensor, with components

$$s_{x1}s_{y2} + s_{y1}s_{x2}; \quad s_{y1}s_{z2} + s_{z1}s_{y2}; \quad s_{z1}s_{x2} + s_{x1}s_{z2};$$

$$s_{x1}s_{x2} - s_{y1}s_{y2}; \quad s_{x1}s_{z2} + s_{y1}s_{y2} - 2s_{z1}s_{z2}.$$

The **first two of these**,  $(i)$  and  $(i')$ , can be used as they stand, cannot be combined with first power expressions of  $p$ , however, since these change sign under the  $t' = -t$  substitution. The last one  $(t)$  gives the familiar expression

$$(i'') \quad (s_1 \cdot r_{12})(s_2 \cdot r_{12}) - 3(s_1 \cdot s_2)r_{12}^2$$

if combined with the similar tensor of the coordinates.<sup>7</sup> It cannot be combined with the  $p$  either. The middle one must be combined with the vector  $p_1 - p_2$  which gives a useless axial invariant and tensor and an ordinary vector. This combined with the distance vector gives the familiar

$$(ia)(ib)(ic) \begin{vmatrix} s_x & s_y & s_z \\ x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ p_{x1} - p_{x2} & p_{y1} - p_{y2} & p_{z1} - p_{z2} \end{vmatrix}.$$

Here  $s_x, s_y, s_z$  can be the components of one of the three vectors  $(v)$ . On the whole, we have 6 invariants. These invariants can be multiplied with one of the six expressions in  $\tau$  which commute with  $\tau_{\xi 1} + \tau_{\xi 2}$ . These are, first of all

$$(\tau_0) \quad 1 \quad \text{and} \quad (\tau_0') \quad -\frac{1}{2} - \frac{1}{2}(\tau_1 \cdot \tau_2) = P_{12}Q_{12}$$

which give the same interaction between all pairs of particles. In addition to these, we have

$$(\tau_1) \quad \frac{1}{2} + \frac{1}{2}\tau_{\xi 1}\tau_{\xi 2} \quad \text{and} \quad (\tau_2') \quad \frac{1}{2}(\tau_{\xi 1} = \tau_{\xi 2}).$$

The first of these gives ordinary interaction but only between like particles; the second gives a negative interaction for proton pairs, a positive

<sup>7</sup>  $(ii)$  has the property that it is identical with  $Q_{12}(ii)$ . It is an interaction which shows saturation.

for neutron pairs, none for unlike particles. These interactions are symmetric in the particles and can be combined with  $(i)$ ,  $(i')$ ,  $(i'')$  and  $(ia)$ , giving in the whole 16 different forms.

Finally we have

$$(\tau_2) \quad \tau_{\xi 1} - \tau_{\xi 2} \quad \text{and} \quad (\tau_2') \quad \frac{1}{2}(\tau_{\xi 1}\tau_{\eta 2} - \tau_{\eta 1}\tau_{\xi 2}),$$

which can be combined with  $(ib)$  and  $(ic)$  giving 4 more types of interaction.

In approximation (1) we can have only  $(i)(\tau_0)$  and  $(i')(\tau_0')$ , i.e., ordinary and Majorana exchange forces.

In approximation (2), all 8 forms derived from  $(\tau_0)$ ,  $(\tau_0')$  and  $(i)$   $(i')$   $(i'')$  and  $(ia)$ . These are, in addition to the previous ones, spin-spin  $(i'')(\tau_0)$ , spin-orbit  $(ia)(\tau_0)$  ordinary forces, Heisenberg forces  $(i)(\tau_0')$ . Furthermore spin-spin exchange forces  $(i'')(\tau_0')$  and spin-orbit exchange forces  $(ia)(\tau_0')$  of the Heisenberg type. The Majorana exchange forces of these types are identical with the ordinary forces. Finally, we have the spin-exchange forces  $(i')(\tau_0)$  of Bartlett.<sup>3</sup>

In approximation (3) we must permit according to the preceding section, in addition to those of 1, only  $(i)(\tau_1)$  and  $(i)(\tau_1')$ , allowing for different interactions between different kinds of pairs. The coefficient of  $(i)(\tau_1')$  is certainly very small, the proton-proton interaction being very nearly equal to the neutron-neutron interaction.

In approximation (4), all 20 types become possible.

### 3

We next go over to approximation (1), and try to define the analog of the multiplet system. This can be defined in two ways: either by considering the functional dependence of the wave functions on the spins or else by considering their dependence on the space coordinates. We shall first consider the spin function.<sup>8</sup>

The great difference between the ordinary spin and the spin considered here is that we have, for every particle, two spin coordinates  $s$  and  $\tau$ , giving in the whole four different sets of values

<sup>8</sup> The content of this section is based on the fundamental mathematical works of E. Cartan, Bull. Soc. Math. de France **41**, 43 (1913). J. de Math. **10**, 149 (1914); I. Schur, Berl. Ber., pp. 189, 297, 346 (1924) and particularly, H. Weyl, Math. Zs. **23**, 271 (1925). I attempted to compile in this section—often without giving rigorous proofs—those results which suffice for the discussion of the physical problems in question.

$-1, -1; -1, 1; 1, -1; 1, 1$ . Instead of two two-valued spins, one can introduce one four-valued spin  $\eta$ , which has the values 1, 2, 3, 4 for the four different doublets of values of  $s$  and  $\tau$ , respectively. This  $\eta$  plays the same role which the two-valued spin plays in the ordinary spin theory. However, because of the four-valuedness of  $\eta$ , instead of the representations of the two-dimensional unitary group (or the equivalent three-dimensional rotation group), the representations of the four-dimensional unitary group will characterize the multiplet systems.

Since the Hamiltonian does not contain the spin coordinates, any transformation which affects only these, will bring a characteristic function into a characteristic function. We can consider first, the permutations of the  $\eta_i$  and second, simultaneous unitary transformations of all the  $\eta$ :

$$R_u \psi(\eta_1, \dots, \eta_n) = \sum_{\vartheta} u_{\eta_1 \vartheta_1} u_{\eta_2 \vartheta_2} \dots u_{\eta_n \vartheta_n} \times \psi(\vartheta_1, \dots, \vartheta_n). \quad (6)$$

We can first define something analogous to the  $Z$  component of the spin momentum by considering the  $u$ 's of the type

$$u(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = \begin{vmatrix} e^{i\varphi_1} & 0 & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 & 0 \\ 0 & 0 & e^{i\varphi_3} & 0 \\ 0 & 0 & 0 & e^{i\varphi_4} \end{vmatrix}. \quad (7)$$

These operations all commute and, hence, a system of functions of the  $\eta$  can be found, the members of which are merely multiplied by constants if an  $R_u$  with  $u$  of the form (7) is applied to them<sup>9</sup>

$$R^{u(\varphi_1 \varphi_2 \varphi_3 \varphi_4)} F^{\nu_{\mu_1 \mu_2 \mu_3 \mu_4}}(\eta_1 \dots \eta_n) = e^{i(\mu_1 \varphi_1 + \mu_2 \varphi_2 + \mu_3 \varphi_3 + \mu_4 \varphi_4)} F^{\nu_{\mu_1 \mu_2 \mu_3 \mu_4}}. \quad (8)$$

The  $\mu$  must be integers in this equation, they will be called diagonal quantum numbers. The  $\nu$  serves only to discriminate between different functions of the  $\eta$  with the same diagonal quantum numbers.

<sup>9</sup> The  $F_{\mu_1 \mu_2 \mu_3 \mu_4}(\eta_1 \dots \eta_n)$  are zero for every set  $\eta_1 \eta_2 \dots \eta_n$  of the  $\eta$ , except for those sets in which  $\mu_1$  of them have the value 1, exactly  $\mu_2$  of them have the value 2, and  $\mu_3$  of them are 3. Then  $\mu_4$  of them will be equal to 4. Otherwise they can be arbitrary and will still satisfy Eq. (8).

Since  $u(\varphi, \varphi, \varphi, \varphi)$  with four equal  $\varphi$  is only multiplication with  $e^{i\varphi}$ , because of (6),  $R_u$  is multiplication with  $e^{in\varphi}$ . It is, on the other hand, multiplication with  $e^{i(\mu_1\varphi+\mu_2\varphi+\mu_3\varphi+\mu_4\varphi)}$  which shows that all possible systems of diagonal quantum numbers satisfy the equation

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = n, \quad (9)$$

where  $n$  is the number of variables  $\eta$ . There is a simple connection between the diagonal quantum numbers and the  $Z$  component of the spin momentum  $S$ . One obtains it by considering a rotation of the spin coordinates around  $Z$  by  $\varphi$ , the matrix of which is of the form (7) with  $\varphi_1 = \varphi_2 = -\varphi_3 = -\varphi_4 = -\frac{1}{2}\varphi$ . Under the influence of the corresponding  $R_u$ , the wave function will be multiplied by  $e^{iS_z\varphi}$  which gives

$$S_z = \frac{1}{2} \sum_k s_{zk} = \frac{1}{2} (\mu_4 + \mu_3 - \mu_2 - \mu_1). \quad (10a)$$

The  $\frac{1}{2}$  before the  $s_{zk}$  enters because the usual definition of the Pauli-matrices is  $1/2$  of that given in (4).

Similarly, we have

$$T_z = \frac{1}{2} \sum_k \tau_{zk} = \frac{1}{2} (\mu_4 - \mu_3 + \mu_2 - \mu_1) \quad (10b)$$

and we define also a

$$Y_z = \frac{1}{2} \sum_k s_{zk} \tau_{zk} = \frac{1}{2} (\mu_4 - \mu_3 - \mu_2 + \mu_1). \quad (10c)$$

The quantum numbers  $S_z$ ,  $T_z$ ,  $Y_z$  can be called magnetic quantum numbers. They determine, together with  $n$ , the  $\mu$  uniquely. Their importance for spectroscopic considerations is the same as that of the single ordinary magnetic quantum number in atomic spectroscopy; they can be easily found simultaneously for all states of a multiplet.

Several states with different magnetic quantum numbers form sets "multiplets" which always have common energy. These sets contain in atomic spectroscopy *one* state with every magnetic quantum number from a maximum,  $S$ , to  $-S$ . We must find the corresponding sets for four-valued spin.

The multiplet will be denoted by the highest set  $\Lambda_4\Lambda_3\Lambda_2\Lambda_1$  of  $\mu$  which occurs in it. The set  $\Lambda_4\Lambda_3\Lambda_2\Lambda_1$  is called higher than the set  $\mu_4\mu_3\mu_2\mu_1$  if either  $\Lambda_4 > \mu_4$ , or if  $\Lambda_4 = \mu_4$  but  $\Lambda_3 > \mu_3$ , or

finally, if  $\Lambda_4 = \mu_4$ ,  $\Lambda_3 = \mu_3$  but  $\Lambda_2 > \mu_2$ . The reason for several states with different diagonal quantum numbers being united into the same multiplet, is that they are transformed into each other by  $R_u$ , the  $u$  of which have not the form (7). Instead of the  $\Lambda$ , we can use for the characterization of a multiplet also

$$S = \frac{1}{2} (\Lambda_4 + \Lambda_3 - \Lambda_2 - \Lambda_1), \quad (11a)$$

$$T = \frac{1}{2} (\Lambda_4 - \Lambda_3 + \Lambda_2 - \Lambda_1), \quad (11b)$$

$$Y = \frac{1}{2} (\Lambda_4 - \Lambda_3 - \Lambda_2 + \Lambda_1), \quad (11c)$$

which together with  $\Lambda_4 + \Lambda_3 + \Lambda_2 + \Lambda_1 = n$  completely determine the  $\Lambda$ .

The character of the multiplets is for the four-fold spin not as simple as for the twofold spin. While the latter ones can be represented by the points on a line from  $-S$  to  $S$ , the former ones must be represented at least in a three-dimensional space, giving the possible  $S_z$ ,  $T_z$ ,  $Y_z$  values and *their multiplicities*. This is necessary because it is not true any more that every combination of  $S_z T_z Y_z$  occurs only once. The multiplicity of every  $S_z T_z Y_z$  is the same as that of any permutation of these numbers and also that of  $-S_z - T_z - Y_z$ ;  $-S_z T_z - Y_z$ ;  $S_z - T_z - Y_z$ . The figure of the multiplet has, therefore, tetrahedral symmetry in the  $S_z T_z Y_z$  space. Using  $S_z T_z Y_z$  has the advantage over using the  $\Lambda$  that the multiplets for  $n=1, 5, 9, 13$ , are represented by the same figures. The quantum numbers are all half-integers, the occurring  $S_z T_z Y_z$  combinations form a face centered lattice for which  $S_z + T_z + Y_z$  is of the form  $2k-1/2$ . These figures, reflected in any of the planes  $T_z Y_z$ ,  $Y_z S_z$ ,  $S_z T_z$ , give the multiplets existing for  $n=3, 7, 11, \dots$ , the face centered lattice being characterized by  $S_z + T_z + Y_z$  having the form  $2k+1/2$ . The quantum numbers  $S_z$ ,  $T_z$ ,  $Y_z$  are integers for even  $n$ . Their sum is even or  $n=4, 8, 12, \dots$ , odd for  $n=2, 6, 10, \dots$ .

In order to find the figures for the multiplets, one must know how many states with a certain  $\mu_4\mu_3\mu_2\mu_1$  combination are present in the multiplet ( $\Lambda_4\Lambda_3\Lambda_2\Lambda_1$ ). We shall denote this number by

$$\left( \begin{matrix} \Lambda_4\Lambda_3\Lambda_2\Lambda_1 \\ \mu_4\mu_3\mu_2\mu_1 \end{matrix} \right). \quad (12)$$

The calculation of the quantities (12) is im-

portant for the following section also. The simplest interpretation of (12) is obtained by considering the subgroup of the unitary group formed by the matrices (7). The symbol (12) denotes how often the (one-dimensional) representation  $e^{i(\mu_1 \varphi_1 + \mu_2 \varphi_2 + \mu_3 \varphi_3 + \mu_4 \varphi_4)}$  occurs in the representation of the total unitary group which is designated by  $(\Lambda_4 \Lambda_3 \Lambda_2 \Lambda_1)$ .

The symbols (12) are defined only if  $\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 = \mu_4 + \mu_3 + \mu_2 + \mu_1$ ,  $\Lambda_4 \geq \Lambda_3 \geq \Lambda_2 \geq \Lambda_1 \geq 0$  and since a permutation of the  $\mu$  does not change the value of (12), we can assume also  $\mu_4 \geq \mu_3 \geq \mu_2 \geq \mu_1 \geq 0$ . The value of (12) is 0, unless  $\Lambda_4 \geq \mu_4$  since  $\Lambda_4$  was the greatest  $\mu$  of the multiplet. In addition to this, (12) vanishes unless<sup>10</sup>

$$\begin{aligned} \Lambda_4 &\geq \mu_4; \quad \Lambda_4 + \Lambda_3 \geq \mu_4 + \mu_3; \\ \Lambda_4 + \Lambda_3 + \Lambda_2 &\geq \mu_4 + \mu_3 + \mu_2. \end{aligned} \quad (13)$$

The last of these can be written also as  $\Lambda_1 \leq \mu_1$ . There are several ways of evaluating (12). One of them is to consider the matrices  $\mu$  which correspond to a three-dimensional unitary transformation of the  $\eta$  values 1, 2, 3 only. The representation  $(\Lambda_4 \Lambda_3 \Lambda_2 \Lambda_1)$  contains all those representations  $(\Lambda_3' \Lambda_2' \Lambda_1')$  of the three dimensional unitary group exactly once for which

$$\Lambda_4 \geq \Lambda_3' \geq \Lambda_3 \geq \Lambda_2' \geq \Lambda_2 \geq \Lambda_1' \geq \Lambda_1. \quad (14)$$

Thus the quartets of diagonal quantum numbers of  $(\Lambda_4 \Lambda_3 \Lambda_2 \Lambda_1)$  are those of all  $(\Lambda_3' \Lambda_2' \Lambda_1')$  satisfying (14), together with the last diagonal quantum number  $\Lambda_4 + \Lambda_3 + \Lambda_2 + \Lambda_1 - \Lambda_3' - \Lambda_2' - \Lambda_1'$ . After this, we can reduce the representation of the three-dimensional unitary group to a two-dimensional: in  $(\Lambda_3' \Lambda_2' \Lambda_1')$  those  $(\Lambda_2'' \Lambda_1'')$  will occur for which

$$\Lambda_3' \geq \Lambda_2'' \geq \Lambda_2' \geq \Lambda_1'' \geq \Lambda_1'. \quad (14a)$$

Finally  $(\Lambda_2'' \Lambda_1'')$  contains the pairs of diagonal quantum numbers  $\Lambda_2'', \Lambda_1''$ ;  $\Lambda_2'' - 1, \Lambda_1'' + 1$ ;  $\Lambda_2'' - 2, \Lambda_1'' + 2$ ;  $\dots$ ;  $\Lambda_1'', \Lambda_2''$ .

For instance, in order to find the multiplet  $(3 \ 1 \ 1 \ 0)$  we can calculate  $(3 \ 1 \ 1 \ 0) = (3 \ 1 \ 1 \ 0) + (2 \ 1 \ 1)1 + (1 \ 1 \ 1)2 + (3 \ 1 \ 0)1 + (2 \ 1 \ 0)2 + (1 \ 1 \ 0)3$ . It suffices to obtain those quadruplets of  $\mu$  which are in a descending order. The others can be

<sup>10</sup> Similar formulas hold also for symbols of the kind (12) with more than four  $\Lambda$ 's. They can be proved by an argument similar to that of the next section.

obtained then by permutation. We can omit hence the underlined ones. To reduce further

$$\begin{aligned} (3 \ 1 \ 1) &= (3 \ 1)1 + (2 \ 1)2 + \underline{(1 \ 1)3} \\ (2 \ 1 \ 1) &= (2 \ 1)1 + \underline{(1 \ 1)2}, \\ (3 \ 1 \ 0) &= \underline{(3 \ 1)0} + (2 \ 1)1 + \underline{(1 \ 1)2} + (3 \ 0)1 \\ &\quad + \underline{(2 \ 0)2} + \underline{(1 \ 0)3}. \end{aligned}$$

This gives the  $\mu$  systems  $3 \ 1 \ 1 \ 0$ ,  $2 \ 2 \ 1 \ 0$ ,  $2 \ 1 \ 1 \ 1$ ,  $2 \ 1 \ 1 \ 1$ ,  $2 \ 1 \ 1 \ 1$  and their permutations. In the language of the magnetic quantum number expressed, the multiplet  $(S, T, Y) = (3/2, 3/2, 1/2)$  contains  $S_z = 3/2$ ,  $T_z = 3/2$ ,  $Y_z = 1/2$  once,  $S_z = 3/2$ ,  $T_z = 1/2$ ,  $Y_z = -1/2$  once,  $S_z = T_z = Y_z = 1/2$  three times. In addition to these, all permutations of these and those triplets in which any two of the  $S_z T_z Y_z$  are replaced by their negative values. The multiplet is shown, along with some other ones, in Fig. 1, it is the third one.

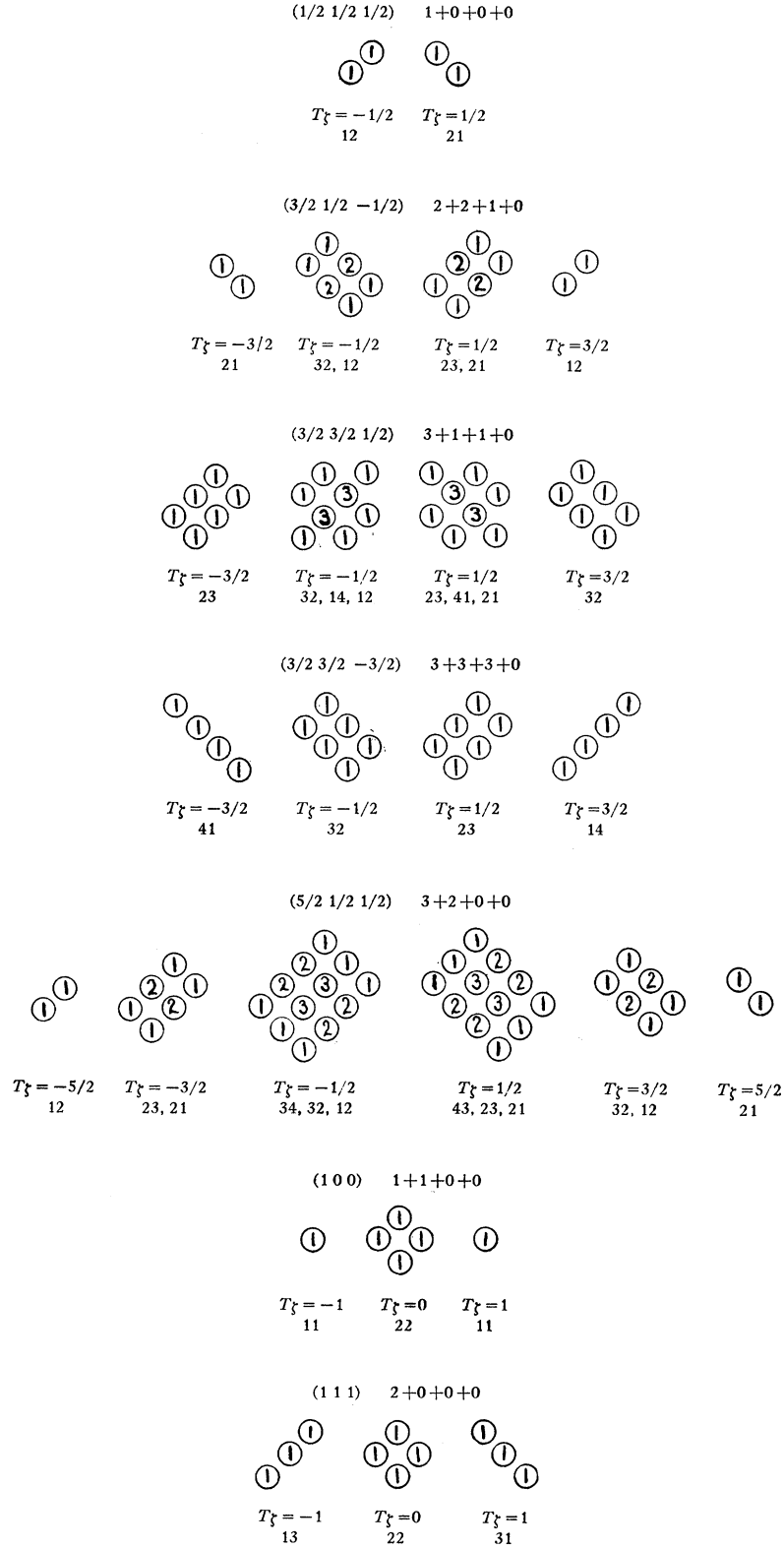
According to the general theory<sup>8</sup> the wave functions of the multiplet  $\Lambda_4 \Lambda_3 \Lambda_2 \Lambda_1$  belong with respect to interchange of the  $\eta$  to the representation of the symmetric group which is characterized by the partition  $\Lambda_4 + \Lambda_3 + \Lambda_2 + \Lambda_1 = n$ .

It has been shown by Slater<sup>11</sup> for atomic spectra that the knowledge of the structure of multiplets enables one to determine the numbers and characters of the terms which arise from any configuration. The same is true in principle for nuclear spectra. The difference is that instead of the two-dimensional plot of occurring  $L_z S_z$  values, one should prepare a four-dimensional plot of  $L_z S_z T_z Y_z$  values. Or perhaps for every  $L_z$  and  $S_z$  a two-dimensional plot of the occurring  $T_z Y_z$  values. For every  $L_z$  these plots must be decomposed into  $S_z T_z Y_z$  combinations which form multiplets. After this, the  $L_z$  values for every multiplet must be grouped together into sets ranging from  $-L$  to  $L$  thus obtaining the azimuthal quantum member.

The most practical procedure along these lines which I could find was one using the  $\Lambda$  and the diagonal quantum numbers. A state with  $\mu_1$  protons and  $\mu_2$  neutrons with spin  $-1/2$  and  $\mu_3$  protons and  $\mu_4$  neutrons with spin  $1/2$  is a state with the diagonal quantum numbers  $\mu_1 \mu_2 \mu_3 \mu_4$ . One first makes a plot of the occurring  $\mu_1 \mu_2 \mu_3$  values for every  $L_z$  and  $\mu_4$ . For this purpose, one draws an equilateral triangle with the altitude

<sup>11</sup> J. C. Slater, Phys. Rev. 34, 1293 (1929).

FIG. 1.



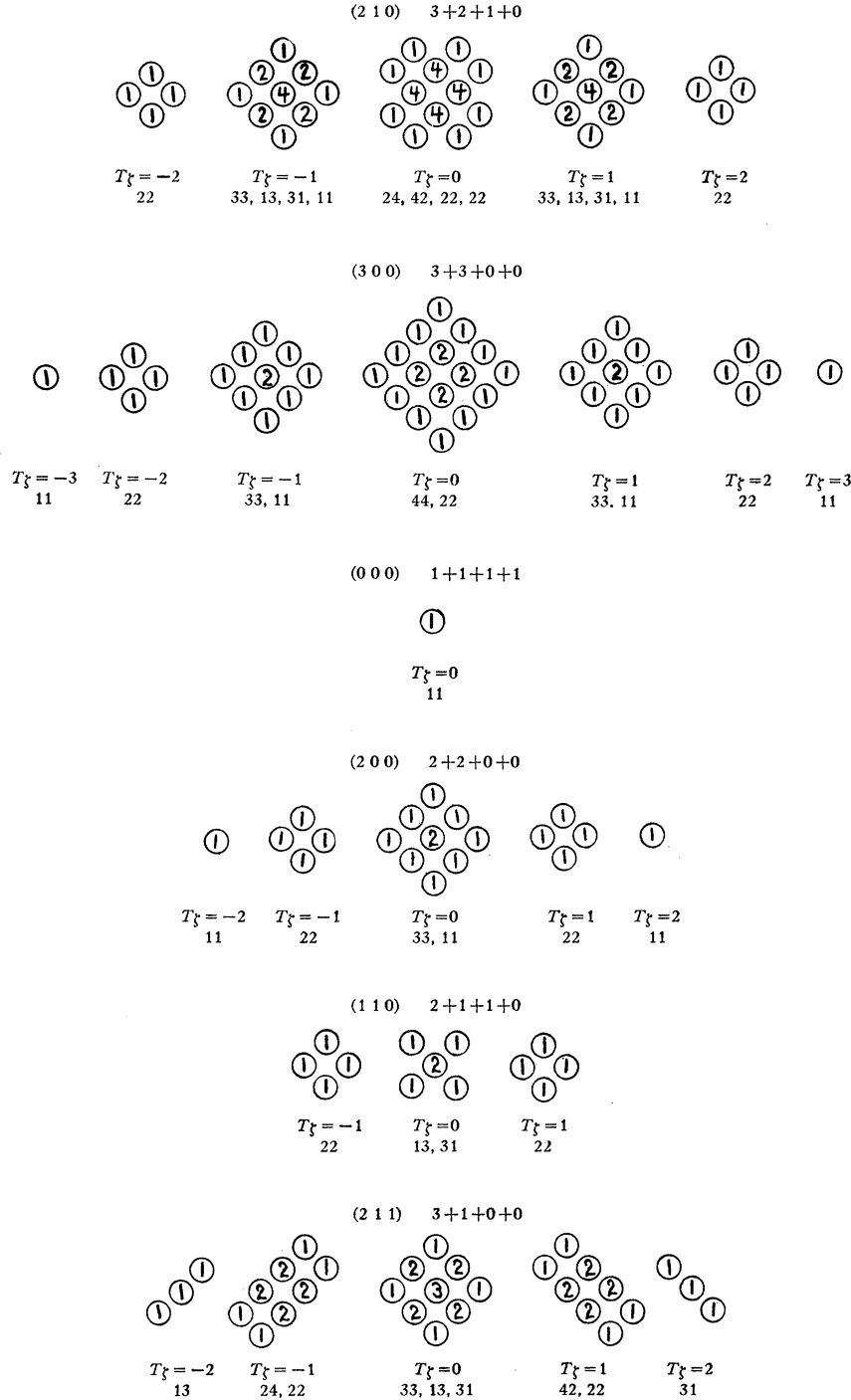


FIG. 1. Every set of figures represents a multiplet, the  $STY$  sign of which, together with one corresponding  $\Lambda_4 + \Lambda_3 + \Lambda_2 + \Lambda_1$ , is given on top. Every circle represents a  $S_z T_Y Y_T$  state,  $T_Y$  is given below the figure.  $Y_T$  and  $S_z$  are the coordinates of the circle, the origin of the coordinate system being at the center of the figure, the  $Y_T$  axis runs to the right, the  $S_z$  axis downward. The numbers in the circles give the number of states  $S_z T_Y Y_T$  in the multiplet. The distance between two adjoining circles on a horizontal or vertical is 2. The multiplets with half integer  $STY$  correspond to elements with masses  $4n+1$ . The multiplets for masses  $4n+3$  are obtained from these by reversing the direction of the  $Y_T$  axis. The sign of  $Y$  must be changed also. The 2 at the center of this figure should be replaced by a 3.



$\mu_1 + \mu_2 + \mu_3 = n - \mu_4$ . The point which has the distances  $\mu_1, \mu_2, \mu_3$  from the three sides of the triangle, respectively, will correspond to a state with the diagonal quantum numbers  $\mu_1, \mu_2, \mu_3$  and  $\mu_4 = n - \mu_1 - \mu_2 - \mu_3$ . Every plot forms a representation of the three dimensional unitary group. The irreducible representations of this group are rather simple, their plots are shown in Figs. 2a and 2b: the combination  $\mu_1\mu_2\mu_3$  occurs in  $\Lambda_1'\Lambda_2'\Lambda_3'$

$$\binom{\Lambda_3'\Lambda_2'\Lambda_1'}{\mu_3\mu_2\mu_1} = 1 + \text{Min}(\Lambda_3' - \mu_3, \Lambda_3' + \Lambda_2' - \mu_3 - \mu_2, \Lambda_3' - \Lambda_2', \Lambda_2' - \Lambda_1') \quad (15)$$

times, where  $\text{Min}(\alpha, \beta, \dots)$  is the smallest of the numbers  $\alpha, \beta, \dots$  if they are positive and equals  $-1$  if any of them is negative.

One can decompose the  $\mu_1\mu_2\mu_3$  plots for every  $L_z$  into irreducible plots, characterized by  $(\Lambda_3'\Lambda_2'\Lambda_1')$  (the primes on the  $\Lambda$  are omitted in the figure). Having obtained the number of  $(\Lambda_3'\Lambda_2'\Lambda_1')$ , one unites these for every  $L_z$  separately into total multiplets  $(\Lambda_4\Lambda_3\Lambda_2\Lambda_1)$  according to

$$(\Lambda_4\Lambda_3\Lambda_2\Lambda_1) = \sum (\Lambda_3'\Lambda_2'\Lambda_1'). \quad (16)$$

The limits of summation are given in (14). Finally the  $L_z$  values for every  $(\Lambda_4\Lambda_3\Lambda_2\Lambda_1)$  are united separately to total azimuthal quantum numbers  $L$ . On the whole, the procedure is much more cumbersome than the analogous one for atomic spectra. It has the disadvantage also, that one first obtains the highest multiplicities, which have the highest energies.

#### 4

An alternative method which leads much more rapidly to the goal is to consider, for the time being, only the dependence of the wave function on space coordinates. This method was worked out for atomic spectra by the present author and Delbrück<sup>12</sup> before the spin theory of Pauli was known. One considers first again the possible configurations i.e. the distributions of the par-

ticles into the different states, without, however, taking into consideration the spin. For every such state one determines the  $L_z$  (or  $m_L$  as it is often called) as the sum of the  $l_z$  of all the particles. Next one determines how many terms with a certain partition  $\lambda_\rho + \lambda_{\rho-1} + \dots + \lambda_1$  this configuration gives. Finally, one considers the states corresponding to every partition separately and unites states with  $L_z$  from  $-L$  to  $L$  into a total multiplet with azimuthal quantum number  $L$ .

These steps performed, one has all one is interested in for approximation (1): the multiplicities and azimuthal quantum numbers of every term. For later work, it is still necessary to know how to complete the wave functions depending on coordinates alone, by functions depending on the spins, to total wave functions. Since the total wave function must be anti-symmetric, the space-coordinate wave function and the spin function used for the completion must have adjoint characters, i.e., belong to two such representations the matrices of which are equal for even, oppositely equal for odd permutations. The adjoint partition to the partition  $4+4+\dots+1$  with  $n_4$  fours,  $n_3$  threes,  $n_2$  twos and  $n_1$  ones is the partition  $\Lambda_4+\Lambda_3+\Lambda_2+\Lambda_1$  where

$$\begin{aligned} \Lambda_4 &= n_4 + n_3 + n_2 + n_1, & \Lambda_3 &= n_4 + n_3 + n_2, \\ \Lambda_2 &= n_4 + n_3, & \Lambda_1 &= n_4. \end{aligned} \quad (17)$$

Thus, e.g., the partition  $4+3+2+2+2+1$  for the spacial wave function is equivalent to the partition  $6+5+2+1$  for the  $\Lambda$ , or to the  $STY$  set  $(4\ 1\ 0)$ .

The step which involves the difficulty for this procedure is to determine how many terms with a certain partition  $\lambda_\rho + \lambda_{\rho-1} + \dots + \lambda_1$  a configuration gives in which there are  $\mu_1$  particles in the first,  $\mu_2$  in the second, etc.,  $\mu_\nu$  in the  $\nu$ -th state. We consider the wave function

$$\psi_1(x_1)\psi_1(x_2)\dots\psi_1(x_{\mu_1})\psi_2(x_{\mu_1+1})\dots\psi_2(x_{\mu_1+\mu_2})\dots\psi_3(x_{\mu_1+\mu_2+1})\dots\psi_\nu(x_n) \quad (18)$$

together with those arising from (18) by a permutation of the  $x$ . There are  $n!/\mu_1!\mu_2!\dots\mu_\nu!$  of these. Under a permutation of the  $x$  they naturally transform among themselves, the corresponding (reducible) representation of the sym-

<sup>12</sup> E. Wigner, Zeits. f. Physik **43**, 627 (1927); M. Delbrück, Zeits. f. Physik **51**, 181 (1928). Eq. (19) of the former and (14) of the latter give an explicit expression for (20) in the case all  $\lambda$  and  $\mu$  are 1 or 2. Only this case occurs in ordinary spectroscopy. A similar expression would be too complicated if the  $\lambda$  and  $\mu$  can be 3 and 4 also.

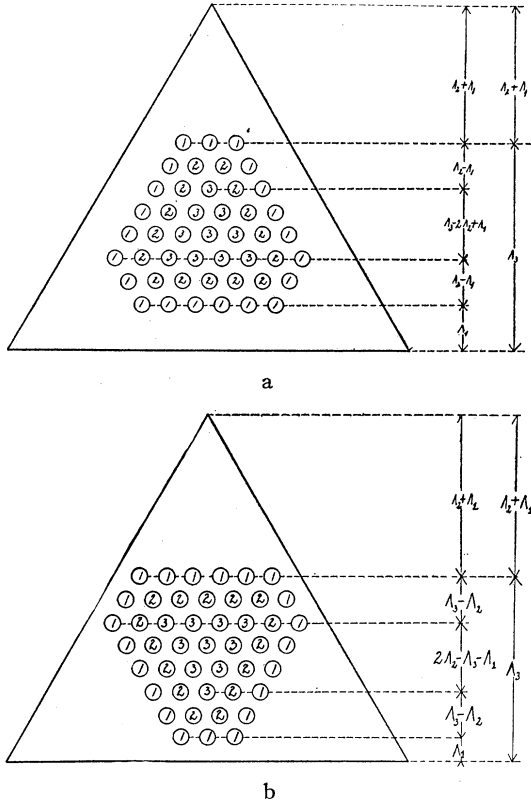


FIG. 2. Every circle corresponds to a triple  $\mu_1\mu_2\mu_3$ , these numbers being the distances of the circle from the three sides of the equilateral triangle, surrounding the figure. The number in the circle tells how often the corresponding set of  $\mu$ 's occurs in the representation characterized by  $(\Lambda_3\Lambda_2\Lambda_1)$ . Fig. 2a holds for  $\Lambda_3-\Lambda_2 \geq \Lambda_2-\Lambda_1$ , Fig. 2b for  $\Lambda_3-\Lambda_2 \leq \Lambda_2-\Lambda_1$  (the first is actually the multiplet (9 4 2) the second one (8 6 1)). In both cases the boundary hexagon contains one's, the next two's and so on until the hexagon reduces to a triangle. The  $\mu$  sets within the triangle occur all equally often in the representation:  $\Lambda_2-\Lambda_1+1$  times in the first,  $\Lambda_3-\Lambda_2+1$  times in the second case.

metric group will be denoted by  $[\mu_\nu\mu_{\nu-1}\cdots\mu_1]$ . Upon decomposing this into irreducible representations

$$[\mu_\nu\mu_{\nu-1}\cdots\mu_1] = \sum_{\lambda} \left( \begin{matrix} \lambda_p & \lambda_{p-1} & \cdots & \lambda_1 \\ \mu_\nu & \mu_{\nu-1} & \cdots & \mu_1 \end{matrix} \right) (\lambda_p\lambda_{p-1}\cdots\lambda_1), \quad (19)$$

the coefficients (some of which occurred previously in (12)) tell us how many terms with the partition  $\lambda_p+\lambda_{p-1}+\cdots+\lambda_1$  the configuration gives. We shall arrange the  $\mu$  again in a descending order  $\mu_\nu \geq \mu_{\nu-1} \geq \cdots \geq \mu_1 \geq 0$ .

Only those partitions  $\lambda_p+\lambda_{p-1}+\cdots+\lambda_1$  will correspond to real terms in which none of the  $\lambda$  is greater than 4. This must hold then, because of

(13), for the  $\mu$  as well: no orbit can be more than four times occupied. One can even omit for the calculation of the coefficients all the fourfold occupied states, i.e., drop all the  $\mu=4$ , since because of (13), the  $\lambda$  above a 4 must be a 4 also.

In the  $p$  shell, there are only three states  $l_z = -1, 0, 1$  and the coefficients of interest are therefore of the form (15) (if the partition contains only two addends, a 0 can be affixed for  $\lambda_1$ ) and they are explicitly given in (15). One must only arrange the  $\mu$  in descending order.

In this case the calculation is especially simple. For instance, for 3 particles we have the ten configurations of Table I. The figures below the  $l_z$  values give the number of particles ( $\mu$ ) in this state, the figure below  $L_z$  is the total  $L_z$ , the last columns give the number of terms with the partitions 3 or (3+0+0) and 2+1 (i.e., (2+1+0)) which this configuration gives. This gives  $F$  and  $P$  terms with the partition (3) and  $D$  and  $P$  terms with the partition (2+1). Table II, reference 4, was prepared in this way. The adjoint partitions to (3) and (2+1) are (1+1+1) and (2+1), respectively, the  $STY$  characterization is  $(\frac{1}{2} \frac{1}{2} -\frac{1}{2})$  for the former,  $(\frac{3}{2} \frac{1}{2} \frac{1}{2})$  for the latter. There is in addition to these, one  $S$  term of the multiplicity  $(\frac{3}{2} \frac{3}{2} \frac{3}{2})$ .

In the general case the explicit formulas for the

$$\left( \begin{matrix} \lambda_p & \lambda_{p-1} & \lambda_{p-2} & \cdots & \lambda_1 \\ \mu_\nu & \mu_{\nu-1} & \mu_{\nu-2} & \cdots & \mu_1 \end{matrix} \right) \quad (20)$$

are too complicated. A useful way of evaluating (20) starts from another interpretation of (20) than given in (19). For Frobenius' reciprocity theorem<sup>13</sup> one considers the subgroup which contains the permutations of the first  $\mu_1$  elements

TABLE I. The ten configurations for 3 particles.

$l_z = -1 \ 0 \ 1$	$L_z$	(3) (2+1)	$l_z = -1 \ 0 \ 1$	$L_z$	(3) (2+1)
3 0 0	-3	1	1 0 2	1	1 1
2 1 0	-2	1 1	0 3 0	0	1 —
2 0 1	-1	1 1	0 2 1	1	1 1
1 2 0	-1	1 1	0 1 2	2	1 1
1 1 1	0	1 2	0 0 3	3	1 —

<sup>13</sup> G. Frobenius, Berl. Ber. 501 (1898). The reader will find a straight forward proof in H. Weyl's *Gruppentheorie und Quantenmechanik* (Leipzig 1928), first edition, p. 254. The proof in the English translation by H. P. Robertson (London, 1931), pp. 332-338, is more abstract.

among themselves, the next  $\mu_2$  elements among themselves, etc. and all the products of these permutations. Then (20) gives the number, how often the unit representation in which every element of the subgroup is represented by the "matrix" (1), occurs in  $(\lambda_\rho + \lambda_{\rho-1} + \dots + \lambda_1)$ , if this is considered as a representation of the subgroup. This is, however, also the number of times the adjoint representation  $(\Lambda_4 \Lambda_3 \Lambda_2 \Lambda_1)$ , again considered as a representation of the sub-

group, contains the "antisymmetric" representation, in which every even permutation is represented by (1), every odd by  $(-1)$ . If we denote this by

$$\begin{bmatrix} \Lambda_4 \Lambda_3 \Lambda_2 \Lambda_1 \\ \mu_\nu \mu_{\nu-1} \dots \mu_1 \end{bmatrix} = \begin{pmatrix} \lambda_\rho \lambda_{\rho-1} \dots \lambda_1 \\ \mu_\nu \mu_{\nu-1} \dots \mu_1 \end{pmatrix} \quad (21)$$

(21) expresses Frobenius' theorem. The  $\Lambda$  are defined in (17).

The expressions (21) can be calculated by recursion formulas. If  $\mu_\nu = 4$ , we have

$$\begin{bmatrix} \Lambda_4 \Lambda_3 \Lambda_2 \Lambda_1 \\ 4 \quad \mu_{\nu-1} \dots \mu_1 \end{bmatrix} = \begin{bmatrix} \Lambda_4 - 1 & \Lambda_3 - 1 & \Lambda_2 - 1 & \Lambda_1 - 1 \\ \mu_{\nu-1} & \mu_{\nu-2} \dots & \dots & \mu_1 \end{bmatrix} \quad (22a)$$

By means of this formula, one can get rid of all 4 among the  $\mu$ . If  $\mu_\nu = 3$

$$\begin{aligned} \begin{bmatrix} \Lambda_4 \Lambda_3 \Lambda_2 \Lambda_1 \\ 3 \quad \mu_{\nu-1} \dots \mu_1 \end{bmatrix} &= \begin{bmatrix} \Lambda_4 - 1 & \Lambda_3 - 1 & \Lambda_2 - 1 & \Lambda_1 \\ \mu_{\nu-1} & \mu_{\nu-2} \dots & \dots & \mu_1 \end{bmatrix} + \begin{bmatrix} \Lambda_4 - 1 & \Lambda_3 - 1 & \Lambda_2 & \Lambda_1 - 1 \\ \mu_{\nu-1} & \mu_{\nu-2} \dots & \dots & \mu_1 \end{bmatrix} \\ &+ \begin{bmatrix} \Lambda_4 - 1 & \Lambda_3 & \Lambda_2 - 1 & \Lambda_1 - 1 \\ \mu_{\nu-1} & \mu_{\nu-2} \dots & \dots & \mu_1 \end{bmatrix} + \begin{bmatrix} \Lambda_4 & \Lambda_3 - 1 & \Lambda_2 - 1 & \Lambda_1 - 1 \\ \mu_{\nu-1} & \mu_{\nu-2} \dots & \dots & \mu_1 \end{bmatrix}. \end{aligned} \quad (22b)$$

After a sufficient number of reductions of this type, there will be only 2's and 1's among the  $\mu$ . If one of the  $\begin{bmatrix} \end{bmatrix}$ , occurring in the right side of (22b) contains a negative number in the upper row, or a number which is greater than the preceding one, the whole  $\begin{bmatrix} \end{bmatrix}$  is zero. To get rid of the 2's

$$\begin{aligned} \begin{bmatrix} \Lambda_4 \Lambda_3 \Lambda_2 \Lambda_1 \\ 2 \quad \mu_{\nu-1} \dots \mu_1 \end{bmatrix} &= \begin{bmatrix} \Lambda_4 & \Lambda_3 & \Lambda_2 - 1 & \Lambda_1 - 1 \\ \mu_{\nu-1} & \mu_{\nu-2} \dots & \dots & \mu_1 \end{bmatrix} + \begin{bmatrix} \Lambda_4 & \Lambda_3 - 1 & \Lambda_2 & \Lambda_1 - 1 \\ \mu_{\nu-1} & \mu_{\nu-2} \dots & \dots & \mu_1 \end{bmatrix} + \begin{bmatrix} \Lambda_4 - 1 & \Lambda_3 & \Lambda_2 & \Lambda_1 - 1 \\ \mu_{\nu-1} & \mu_{\nu-2} \dots & \dots & \mu_1 \end{bmatrix} \\ &+ \begin{bmatrix} \Lambda_4 & \Lambda_3 - 1 & \Lambda_2 - 1 & \Lambda_1 \\ \mu_{\nu-1} & \mu_{\nu-2} \dots & \dots & \mu_1 \end{bmatrix} + \begin{bmatrix} \Lambda_4 - 1 & \Lambda_3 & \Lambda_2 - 1 & \Lambda_1 \\ \mu_{\nu-1} & \mu_{\nu-2} \dots & \dots & \mu_1 \end{bmatrix} + \begin{bmatrix} \Lambda_4 - 1 & \Lambda_3 - 1 & \Lambda_2 & \Lambda_1 \\ \mu_{\nu-1} & \mu_{\nu-2} \dots & \dots & \mu_1 \end{bmatrix}. \end{aligned} \quad (22c)$$

If all the  $\mu = 1$ , which will be true after some reductions of this type, one can use a formula analogous to the previous ones. It is quicker to notice that (23) is the dimension of  $(\Lambda_4 \Lambda_3 \Lambda_2 \Lambda_1)$  and hence

$$\begin{aligned} \begin{bmatrix} \Lambda_4 \Lambda_3 \Lambda_2 \Lambda_1 \\ 11 \dots 1 \end{bmatrix} &= \frac{(\Lambda_4 + \Lambda_3 + \Lambda_2 + \Lambda_1)! (\Lambda_4 + 3 - \Lambda_1) (\Lambda_4 + 2 - \Lambda_2) (\Lambda_4 + 1 - \Lambda_3)}{(\Lambda_4 + 3)! (\Lambda_3 + 2)! (\Lambda_2 + 1)! \Lambda_1!} \\ &\cdot (\Lambda_3 + 2 - \Lambda_1) (\Lambda_3 + 1 - \Lambda_2) (\Lambda_2 + 1 - \Lambda_1). \end{aligned} \quad (23)$$

One will use the formulas (22), (23) very rarely. In most practical cases, the calculation of the coefficients (21) is greatly facilitated by special conditions.

5

We can go over, now, to approximation (2). In this approximation the Hamiltonian will be

invariant with respect to all operations involving  $\tau$  only. Since  $\tau$  has, mathematically, the same properties as the ordinary spin variable, we shall have the analogs of the quantum numbers  $L$  and  $S$  of ordinary spectroscopy. Instead of the azimuthal quantum number of the common spectroscopy, we have the total quantum number  $J$ , since the spin forces are taken into account in

approximation 2; instead of the total spin, we have an isotopic spin  $T$ . Of course, the interaction of  $J$  and  $T$  in the higher approximation 4 will be entirely different from the interaction between azimuthal and spin quantum numbers of ordinary spectroscopy. Above all, the resulting angular momentum  $J$  will be a good quantum number in all approximations.

The existence of a total isotopic spin means that terms with different  $\zeta$  components of the isotopic spin have the same energy in approximation 2. These are, of course, terms of different isobaric nuclei, and a total isotopic spin  $T$  will be a term with the same binding energy for all nuclei with isotopic numbers from  $-T$  to  $T$ . This shows that to every term of an element with a certain isotopic number  $T_\zeta$ , terms of all elements with smaller isotopic numbers will correspond. The element with the smallest isotopic number ( $n_P = n_N$  for even masses,  $n_P = n_N \pm 1$  for odd masses) has the greatest number of terms. In approximation 4, the equality of these term values will cease to hold and the Coulomb energy, already, will introduce a splitting.

If one is interested in the number of terms of approximation (2), arising from a certain configuration, one can use the ordinary Hund-Russell-Saunders method to determine these. The only difference is that the "orbits" contain the ordinary spin quantum number already and one has, therefore, for instance, six  $p$  states, with  $Z$  components of the angular momentum  $3/2, 1/2, 1/2, -1/2, -1/2, -3/2$ . Everyone of these six states can be doubly occupied, with a particle  $\tau = 1$  and  $\tau = -1$  (neutron or proton). The half sum of the  $\tau$  is denoted by  $T_\zeta$  and the different  $T_\zeta$  from  $-T$  to  $T$  united into a multiplet. The number of terms, arising even from a simple configuration, is very great, however.

It is more important, perhaps, to consider the terms into which a term of approximation (1) splits if we introduce the spin forces and thus go over to (2). The transition from approximation (1) to (2) can be performed in two steps: first disregarding  $Y_\zeta$ , every multiplet goes over into several multiplets which still have an  $S$  and  $T$ . One obtains these by simply projecting every point of Fig. 1 into the  $S_z T_\zeta$  plane. This is done in Fig. 3 for the multiplet  $(3/2, 3/2, 1/2)$  as an example. We see that it gives one term with

$S = 3/2, T = 3/2$ , one with  $S = 3/2, T = 1/2$ , one with  $S = 1/2, T = 3/2$ , and one with  $S = 1/2, T = 1/2$ . The second step, then, is to combine the  $S$ 's with the azimuthal quantum number  $L$  to  $J$ 's, according to the vector addition model.

It would be very important to know experimentally the relative separation of the terms which arise from the same approximation 1 term, since this would allow us to tell which of the 6 possible interactions, given in Section 2, describes the spin forces.

Fig. 4 shows what can be expected in approximation (2). Every figure corresponds to a set of isobars. The abscissa is the isotopic number, the ordinate the total energy. Every line corresponds to a term of approximation (2) all lines arising from the same term of approximation (1) are grouped close together. The  $STY$  symbol of this term is given on the right, it is, of course, the same for all the group. The  $T$  of the term is represented by the length of its line, so that the term exists for elements with those isotopic numbers  $T_\zeta$ , over which the line extends. The number on the left of the line is the  $S$ , characterizing its spin after the first step in the transition from approximation (1) to (2) is performed. This  $S$  will be the total angular momentum  $J$  of the nucleus, if the azimuthal quantum number was zero, which will be very frequently the case. The energy of the approximation (1) term is estimated on the basis of Eq. (8), reference 4. This estimate gives the same value for all terms of the same configuration with the same multiplicity  $STY$  which is, of course, only approximately true. We are interested, of course, only in the lowest term of every multiplicity. The distances between the lines of the same group have no significance.

We see that in several cases the approximation (2) terms extend over several isobars and the question of the most stable isobar will be decided, hence, only in the next approximation. We may assume that the most important term in the next approximation is the Coulomb energy.<sup>14</sup> This will

<sup>14</sup> The most stable isobar has the smallest mass, not the greatest binding energy. For the consideration of the stability, therefore,  $T_\zeta$  times the mass difference between neutron and  $H^1$  should be added to the total energy. This will cause the lines of Fig. 3 to slope upward to the right. This slope is soon overcompensated, however, by the opposite effect of the Coulomb energy.

$T_z = -\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	
$S_z = -\frac{3}{2}$	(1)	(2)	(2)	(1)
$-\frac{1}{2}$	(2)	(4)	(4)	(2)
$\frac{1}{2}$	(2)	(4)	(4)	(2)
$\frac{3}{2}$	(1)	(2)	(2)	(1)

FIG. 3. Determination of the spin angular momentum  $S$  and isotopic spin  $T$  of the terms arising from the multiplet  $(3/2, 3/2, 1/2)$  if the spin forces are introduced. The spin angular momenta  $S$  must be added, subsequently, to the orbital angular momentum  $L$ , according to the vector addition principle, in order to obtain the total angular momentum  $J$ .

decrease the binding energy of the nuclei with negative  $T_z$ , compared with the binding energy of nuclei with positive  $T_z$  and cause the horizontal lines of Fig. 4 to slope downward to the right. The slope will be very roughly proportional to the  $2/3$  power of the charge.

This slope will have the most noticeable effect for isobars with masses  $4n+2$ . While for small charge, the point  $a$  is most stable, beginning at  $O^{18}$ , the point  $b$  will become stable.

We can proceed even to higher elements, by successively increasing the slope of the lines more and more. For elements  $4n$ , if the slope becomes  $3/2$ , in the arbitrary units of the figure, the point  $b$  will become most stable. This happens to be at  $A^{40}$ . The point  $c$  never will become most stable, since  $b$  reaches  $a$  before  $c$  does. This seems to be the explanation why no nuclei of mass  $4n$  with odd number of protons and neutrons exist. There are, however, radioactive nuclei of this type.

The situation is very similar for nuclei with masses  $4n+2$ . Here the critical slope is 2, when the point  $c$  reaches  $a$ . Again, point  $d$  is not the most stable for any slope, and there are (apart from  $a$ ) no stable nuclei with odd neutron and proton number, for elements  $4n+2$  either. The slope 2 seems to be reached at  $Ti^{50}$ , later, of course, than slope  $3/2$ .

It should be mentioned that the whole Fig. 4 will be compressed in energy scale as we proceed to higher elements, because the exchange integrals decrease. It has been shown by Bethe and

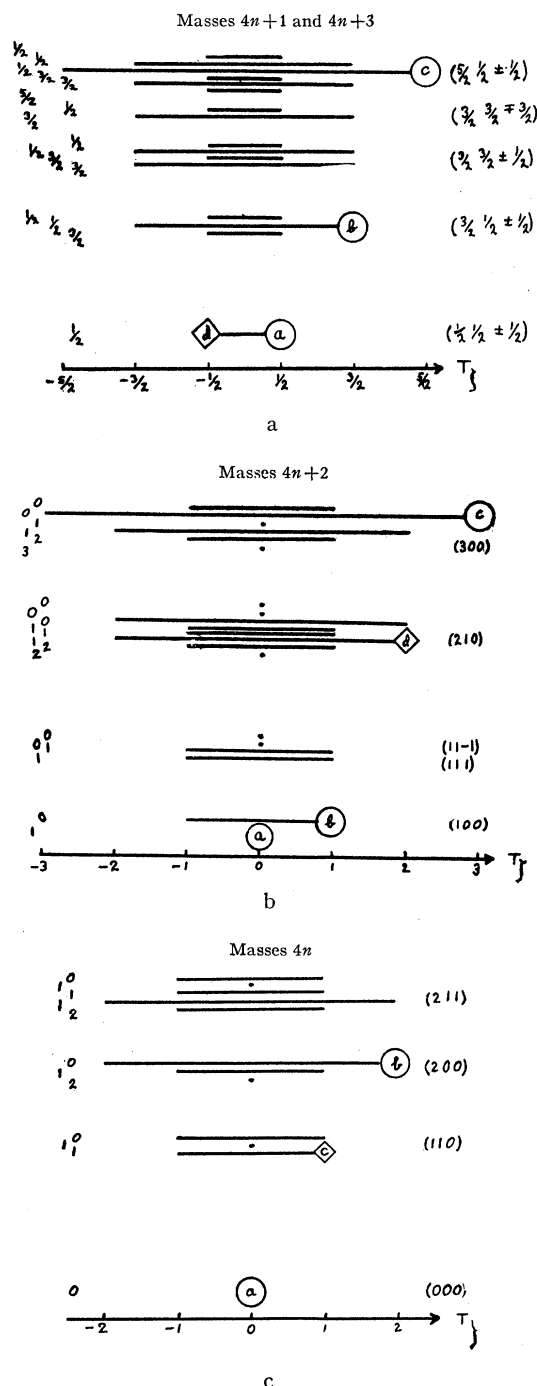


FIG. 4. The different kinds of multiplets are shown for elements with mass numbers of the form  $4n$ ,  $4n+1$ ,  $4n+2$ . The ordinate is the energy in arbitrary units. Only one term for every multiplet system is given, with a position on the energy scale corresponding to long range forces. The abscissa is the difference between the number of neutrons and protons, divided by 2. The circles correspond to stable nuclei, the squares to unstable nuclei.

Bacher<sup>15</sup> that this happens in discontinuous steps, corresponding to the completion of shells. There is, therefore, an increased probability for the slope to pass a certain amount in the regions where shells are completed.

In case of elements with mass numbers  $4n+1$  and  $4n+3$  we obtain the same picture. First, the point  $a$  is most stable, at a slope  $3/2$  the point  $b$  reaches  $a$ . This seems to happen at  $\text{Cl}^{37}$  and  $\text{Ca}^{43}$ , respectively, quite in the neighborhood of  $A^{40}$ . For the slope 2, the point  $c$  passes  $b$  and  $\text{Ti}^{49}$  and  $\text{V}^{51}$  become the stable isobars. These are near indeed to  $\text{Ti}^{50}$ . This explanation of the places where the isotopic number of stable isobars shifts to higher values works rather better than could be expected and the agreement is beyond doubt, partly accidental.

## 6

As a last point, I should like to establish the connection between approximations 1 and 3, i.e., determine the terms into which an  $STY$  term of approximation (1) splits if one introduces, as a perturbation, the difference between proton-proton, proton-neutron, neutron-neutron interactions, neglecting, however, spin forces.

The operator

$$\frac{1}{4}(s_{z1}(1+\tau_{\zeta 1})+s_{z2}(1+\tau_{\zeta 2})+\dots +s_{zn}(1+\tau_{\zeta n}))=\frac{1}{2}(S_z+Y_{\zeta})=S_{zN} \quad (24)$$

gives the  $Z$  component of the neutron spin angular momentum, since  $1+\tau_{\zeta}$  gives 0, if applied to a proton state. Similarly

$$\frac{1}{4}(s_{z1}(1-\tau_{\zeta 1})+s_{z2}(1-\tau_{\zeta 2})+\dots +s_{zn}(1-\tau_{\zeta n}))=\frac{1}{2}(S_z-Y_{\zeta})=S_{zP} \quad (24a)$$

gives the  $Z$  component of the proton spin angular

momentum. If we go through all the points of Fig. 1 for a certain  $T_{\zeta}$  and insert their  $S_{zN}$  and  $S_{zP}$  values into a table, one can unite the points of the table in the normal way to a  $S_N S_P$  multiplet.<sup>16</sup> The azimuthal quantum number is, of course, unchanged by the transition from approximation (1) to (3).

If the proton-proton forces are assumed to be equal to the neutron-neutron forces, the  $S_N S_P$  term will coincide in case of equal number of protons and neutrons with the  $S_P S_N$  term, if  $S_N \neq S_P$ . In the work of reference 4, these terms were given as one term. The values of the multiplicities  $2S_N+1$ ,  $2S_P+1$  are given in Fig. 1 below every  $T_{\zeta}$ .

We may consider, as an example, the  $(3/2 \ 3/2 \ 1/2)$  multiplet, given in Fig. 1. Below  $T_{\zeta}=1/2$  we have the three pairs 23, 41, 21. This means that for an element of the mass  $4n+1$  with  $T_{\zeta}=1/2$  (e.g.,  $\text{C}^{13}$ ), certain states in which the neutrons are in the doublet, the protons in the triplet state, exactly coincides in approximation (1) with a state in which the neutrons are in the quartet, the protons in the singlet state, and with another state in which the neutrons are in doublet, the protons in the singlet state. It may be added that the binding energy of these states is equal in approximation (1), to the binding energy of a hypothetical  $\text{B}^{13}$  nucleus ( $T_{\zeta}=3/2$ ) in which the neutrons are in the triplet, the protons in the doublet state. In reality, all these states will be unstable for isobars with the mass 13, because of the comparatively high position of the  $(3/2 \ 3/2 \ 1/2)$  multiplet in Fig. 4a. The example is thus, perhaps, not a very fortunate one but it illustrates the kind of regularities to be expected more clearly than a simpler case.

<sup>15</sup> H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. 8, 82 (1936), Section VI.

<sup>16</sup> Instead of this, one can simply turn the corresponding level in the diagrams of Fig. 1 by  $45^\circ$ .